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MODEL UPDATING USING AN INCOMPLETE SET OF EXPERIMENTAL MODES

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New approaches are presented that use the measured natural frequencies and mode shapes to update the analytical mass and stiffness matrices of a structure. By adding known masses to the structure and measuring its new modes of vibration, we can utilize this additional information to correct the mass matrix of the system, after which the stiffness matrix can be updated by requiring it to satisfy the generalized eigenvalue problem associated with the structure. Manipulating the unknown system matrices into vector forms, the connectivity information can be easily implemented to preserve the physical configuration of the structure, and to reduce the computational efforts required to correct the system matrices. A comparison is made between the proposed updating schemes introduced in this paper and other updating algorithms found in the literature, and drastic improvements are observed.

1. INTRODUCTION

Highly accurate and detailed analytical models are required to analyze and predict the dynamical behavior of complex structures. With the advent of digital computers, new methods of analysis have been developed, especially in the method of finite elements. Once the finite element model of a physical system is constructed, it is often validated by comparing its analytical modes of vibration with the results of a model survey. If the model survey and the analytical predictions are in subjective agreement, then more credence is given to the analytical model, and it can be used with more confidence for future analysis. If the correlation between the two is unsatisfactory, then the analyst has the options of accepting the analysis, accepting the tests, modifying either or modifying both.

The lack of correlation between the analytical predictions and the experimental results can be traced to either experimental or modelling errors, or a combination of both. When performing vibration tests, many sources of errors may arise, including inexact equipment calibration, excessive noise, equipment malfunction, misinterpretation of data, incorrect transducer locations, etc. Analytical finite element models may also contain errors, including inappropriate modelling assumptions, uncertainties in the material properties, insufficient modelling details, typographical computer inputs, incorrect boundary conditions, etc. Here we do not address the questions of improving testing methods or improving analysis procedures. We assume the available measured natural frequencies and mode shapes are exact. Thus, when the analytical predictions do not match the test measurements, the finite element model must be corrected or updated such that the

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agreement between predictions and test results is improved. The updated model may then be considered a better dynamical representation of the structure. The above process is known as *model updating*.

In this paper, we will first summarize two commonly referenced updating algorithms found in the literature. The underlying principles for each will be discussed, and the pros and cons will be addressed. We will then propose new model updating schemes to adjust the system mass and stiffness matrices from an incomplete set of measured modes, and discuss the techniques needed to solve the resulting problems. Finally, a comparison will be made between the various model updating algorithms.

2. MODEL UPDATING

In recent years many methods have been developed to improve the quality of the analytical finite element models using test data. Detailed discussion of every approach is beyond the scope of this paper, and interested readers are referred to the recent survey paper by Mottershead and Friswell [1]. In this paper, we will only introduce two commonly referenced model updating techniques.

In most updating algorithms, all the co-ordinates of a given normal mode must be known. Due to physical limitations, time or cost constraints, however, the number of measured co-ordinates is generally substantially less than the degrees of freedom of the analytical model. Thus, before any updating algorithm is implemented, we first have to expand the measured mode shapes to the same size as their analytical counterparts. Mode shape or eigenvector expansion is a key feature in many model updating schemes. This process is known as *mode expansion*. A detailed comparison of various mode expansion methods can be found in reference [2]. Because mode shape expansion introduces errors in all of the updating algorithms considered in this paper, we asume that all of the co-ordinates can be measured. This allows us to compare the updating algorithms themselves and not confound the resulting updates with errors introduced by mode expansion. Thus, the effects of the incompleteness of the measured co-ordinates will not be addressed here, and the effects of the various mode expansion algorithms on the quality of the updates will not be pursued. Finally, in the subsequent analysis, the mass and stiffness matrices of the actual and analytical systems are all symmetric.

2.1. LAGRANGE MULTIPLIERS APPROACH

Berman [3] developed a method that uses the measured mode shapes to correct the mass matrix of a structure. This updating scheme identifies, without iteration, a set of minimum changes in the analytical mass matrix such that the measured modes are orthogonal to the updated mass matrix of the system. Using the Lagrange multipliers formalism to optimally correct the mass matrix subjected to the orthogonality constraint, he derived an expansion for the updated mass matrix, $[M]$, of size $N \times N$ (where N corresponds to the degrees of freedom of the analytical model), as follows:

$$[M] = [M_0] + [M_0][X][m]^{-1}([I] - [m])[m]^{-1}[X]^{\mathrm{T}}[M_0], \qquad (1)$$

where $[M_0]$ is the analytical mass matrix, $[I]$ is the identify matrix, $[X]$ is the experimentally determined rectangular modal matrix, of size $N \times N_e$, where N_e is the number of measured modes, and

$$[m] = [X]^{\mathrm{T}}[M_0][X]. \qquad (2)$$

Matrices $[I]$ and $[m]$ are both of sizes $N_e \times N_e$. For an incomplete set of measured modes, $N_e < N$.

Using essentially the method first introduced by Baruch and Bar Itzhack [4], Wei [5] developed an optimal method to update the stiffness matrix of a structure. He also employed the Lagrange multipliers formalism to update the stiffness matrix subjected to the constraints of satisfying the generalized eigenvalue problem, the orthogonality condition of the measured mode shapes and the symmetry property of the stiffness matrix. He found the updated stiffness matrix $[K]$, to be given by

$$[K] = [K_0] + ([\Delta] + (\Delta]^{\mathrm{T}}), \tag{3}$$

where

$$[\Delta] = \tfrac{1}{2}[M][X]([X]^{\mathrm{T}}[K_0][X] + [\Lambda])[X]^{\mathrm{T}}[M] - [K_0][X][X]^{\mathrm{T}}[M]. \tag{4}$$

Matrix $[K_0]$ is the analytical stiffness matrix, and matrix $[\Lambda]$ is a diagonal matrix whose elements are the measured eigenvalues (natural frequencies squared) of the system. While the Lagrange multipliers formalism updates the system stiffness and mass matrices without iteration, the resulting updated matrices are fully populated, implying that the updated model may introduce masses and load paths that do not physically exist. Thus, certain off-diagonal terms of these matrices are fictitious, and they are artifacts of the updating scheme. Moreover, the resulting updated matrices may suffer a loss of positive difiniteness during the updating process, and the updated model may generate spurious modes in the frequency range of interest [1].

The Lagrange multipliers approach to update the system matrices return fully populated mass and stiffness matrices that bear little resemblance to the physical system being analyzed. To preserve the physical load paths of the original analytical model, Kabe [6] assumed the analytical mass matrix to be correct and incorporated the structural connectivity information, which is generally well known, in addition to the test data to optimally adjust the stiffness matrix. The adjustments he performed are such that zero and non-zero elements of the analytical model are preserved, and the adjusted model exactly reproduces the models used in the identification. He also utilized a Lagrange multipliers formalism, so that the percentage change to each stiffness element is minimized. While Kabe's approach to updating the stiffness matrix is straightforward, the assumption that the actual mass matrix is identical to the analytical mass matrix remains questionable [7]. Moreover, Kabe's approach is limited by the storage required and is very computational intensive, since in order to update the stiffness matrix, a large standard eigenvalue problem of size $NN_e \times NN_e$ needs to be solved [6].

2.2. PERTURBATION APPROACH

Using an approach based on the matrix perturbation theory, Chen *et al.* [8] found the updated mass matrix to be

$$[M] = [M_0] + [M_0][X_0](2[I] - [X_0]^{\mathrm{T}}[M_0][X] - [X]^{\mathrm{T}}[M_0][X_0])[X_0]^{\mathrm{T}}[M_0], \tag{5}$$

where $[X_0]$ is the normalized modal matrix such that

$$[X_0]^{\mathrm{T}}[M_0][X_0] = [I] \tag{6}$$

and $[X]$ is the measured modal matrix. Matrices $[X_0]$ and $[X]$ are both of size $N \times N_e$. Using the same technique, Chen *et al.* [8] also derived the following expression for the updated stiffness matrix:

$$[K] = [K_0] + [M_0][X_0](2[\omega_0^2] + 2[\omega_0][\delta\omega] - [X_0]^{\mathrm{T}}[K_0][X] - [X]^{\mathrm{T}}[K_0][X_0])[X_0]^{\mathrm{T}}[M_0], \tag{7}$$

where $[\omega_0]$ is diagonal matrix whose elements are the analytical natural frequencies associated with $[K_0]$ and $[M_0]$, and $[\delta\omega]$ is a diagonal matrix whose elements correspond to the differences between the measured and the analytical natural frequencies.

While straightforward, the perturbation approach of updating the mass and stiffness matrices processes certain shortcomings. Like the schemes proposed by Berman and Wei, equations (5) and (7) also result in fully populated mass and stiffness matrices, thus failing to preserve the physical connectivity of the system. Moreover, because their method is based on the matrix perturbation theory whereby the second and higher order terms are ignored, the algorithm can only be applied when the analytical and the actual system matrices are close. When the system matrices deviate substantially from one another, the approach introduced in reference [8] leads to an erroneous updated model due to the truncation of higher order terms. Finally, the derivation carried out by Chen *et al.* requires that the measured model $[X]$, satisfy

$$[X]^{\mathrm{T}}[M][X] = [I], \qquad [X]^{\mathrm{T}}[K][X] = [\Lambda]. \tag{8}$$

Because the objective of model updating is to correct the system mass and stiffness matrices $[M]$ and $[K]$ are not known *a priori*. Thus, the necessary orthogonality constraints cannot be enforced, and their proposed model updating approach based on the perturbation theory cannot be utilized in practice.

2.3. PROPOSED MODEL UPDATING ALGORITHM

We will now introduce an alternative model updating technique that is simple to apply and can easily accommodate the connectivity information, which is assumed to be readily available and well known, to preserve the physical configuration of the system. The modes of vibration of the actual system must satisfy the generalized eigenvalue problem

$$[K][X] = [M][X][\Lambda], \tag{9}$$

where $[M]$ and $[K]$ are the actual system matrices (both of sizes $N \times N$), $[X]$ is the measured modal matrix (of size $N \times N_e$) of the system, and $[\Lambda]$ is a diagonal matrix (of size $N_e \times N_e$) whose elements are the measured eigenvalues of the system. Assuming the matrices $[M]$ and $[K]$ can be expressed as

$$[M] = [M_0] + [\delta M], \qquad [K] = [K_0] + [\delta K], \tag{10}$$

where $[\delta M]$ and $[\delta K]$ represent the mass and stiffness correction matrices, respectively, then equation (9) becomes

$$([K_0] + [\delta K])[X] = ([M_0] + [\delta M])[X][\Lambda]. \tag{11}$$

Premultiplying the above equation by $[X]^{\mathrm{T}}$ and expanding the resulting matrix equation, we get the following matrix equation of size $N_e \times N_e$:

$$[\Lambda'] + [X]^{\mathrm{T}}[\delta K][X] = ([I'] + [X]^{\mathrm{T}}[\delta M][X])[\Lambda], \tag{12}$$

where

$$[\Lambda'] = [X]^{\mathrm{T}}[K_0][X], \qquad [I'] = [X]^{\mathrm{T}}[M_0][X]. \tag{13}$$

In general, neither the analytical mass matrix nor the analytical stiffness matrix will be exact. Thus, $[\delta M] \neq [0]$ and $[\delta K] \neq [0]$. For a given set of measured modes, an infinite number of $[\delta M]$ and $[\delta K]$ combination may satisfy equation (12). To introduce another set of matrix equation, we first attach known masses to the system of interest, at locations coincident with the nodal points of the finite element model in order to preserve the size of the initial analytical system, and then measure the modes of vibration of this newly constructed mass-modified system. In conjunction with the original set of experimental data we can readily update the mass matrix of the structure.

The measured modes of the actual system must satisfy equation (9). We now add a known mass matrix $[M_a]$, to the system so that the new system satisfies the generalized eigenvalue problem

$$[K][X_a] = ([M] + [M_a])[X_a][\Lambda_a], \tag{14}$$

where $[X_a]$ corresponds to the $N \times N_e$ modal matrix of the new system, and $[\Lambda_a]$ consists of a diagonal matrix, of size $N_e \times N_e$, whose elements are the eigenvalues of the new system. The stiffness matrix of this mass-modified system is assumed to remain unchanged from the initial structure. Taking the transpose of equation (9) and postmultiplying the resultant matrix equation by $[X_a]$, we get

$$[X]^{\mathrm{T}}[K][X_a] = [\Lambda][X]^{\mathrm{T}}[M][X_a]. \tag{15}$$

Premultiplying equation (14) by $[X]^{\mathrm{T}}$, we have

$$[X]^{\mathrm{T}}[K][X_a] = [X]^{\mathrm{T}}([M] + [M_a])[X_a][\Lambda_a]. \tag{16}$$

Equating the right-hand sides of equations (15) and (16), we obtain

$$[\Lambda][X]^{\mathrm{T}}[M][X_a] = [X]^{\mathrm{T}}([M] + [M_a])[X_a][\Lambda_a]. \tag{17}$$

or

$$[\Lambda][X]^{\mathrm{T}}[M][X_a] - [X]^{\mathrm{T}}[M][X_a][\Lambda_a] = [X]^{\mathrm{T}}[M_a][X_a][\Lambda_a]. \tag{18}$$

Defining

$$[P] = [X]^{\mathrm{T}}[M][X_a], \tag{19}$$

then equation (18) becomes

$$[\Lambda][P] - [P][\Lambda_a] = [Q], \tag{20}$$

where

$$[Q] = [X]^{\mathrm{T}}[M_a][X_a][\Lambda_a]. \tag{21}$$

The (i, j)th element of equation (20) yields

$$(\lambda_i - \lambda_{aj})P_{ij} = Q_{ij}, \tag{22}$$

where λ_{aj} is the jth measured eigenvalue of the mass-modified system, and $i, j = 1, \ldots, N_e$. Assuming that the N_e measured eigenvalues of the original and the mass-modified systems are distinct, then we can solve for all the unknowns P_{ij} and construct the matrix $[P]$.

Finally, masses of any magnitude can be added as long as the resulting mass-modified system and the initial structure have distinct measured eigenvalues, i.e., $\lambda_i \neq \lambda_{aj}$. Numerical simulations indicate that (1) the added masses can be an order of magnitude smaller than the actual masses, and (2) the required number of added masses is only a fraction of the size of the analytical model. Thus, the assumption that added masses will not significantly affect the stiffness of the initial physical structure is valid, and the proposed updating scheme shows promise in actual application.

Equation (19) can also be written as

$$[X]^{\mathrm{T}}[\delta M][X_a] = [P] - [X]^{\mathrm{T}}[M_0][X_a]. \tag{23}$$

Because $[X]$ and $[X_a]$ are both rectangular matrices (assuming $N_e < N$), they have no inverses. However, equation (23) can be expanded such that $[\delta M]$ appears as an unknown column vector $\delta\mathbf{m}$ as follows:

$$[A]\,\delta\mathbf{m} = \mathbf{r}, \tag{24}$$

where

$$\delta\mathbf{m} = [\delta m_{11} \cdots \delta m_{1N} \,|\, \delta m_{21} \cdots \delta m_{2N} \,|\, \cdots \,|\, \delta m_{N1} \cdots \delta m_{NN}]^{\mathrm{T}} \tag{25}$$

and

$$\mathbf{r} = [r_{11} \cdots r_{1N_e} | r_{21} \cdots r_{2N_e} | \cdots | r_{N_e 1} \cdots r_{N_e N_e}]^{\mathrm{T}}. \tag{26}$$

In equation (25), δm_{ij} corresponds to the (i, j)th element of $[\delta M]$. Matrix $[A]$ is of size $N_e^2 \times N^2$, whose elements can be determined by expanding the left-hand side of equation (23), vector $\mathbf{r}$ is of length N_e^2, whose components can be obtained by expanding the right-hand side of equation (23).

When $N_e = N$, we have as many equations as we do unknowns, and equation (24) can be solved exactly by using simple Gauss elimination. When $N_e < N$, equation (24) results in an underdetermined problem (that is, the number of equations is less than the number of unknowns), which typically has an infinite number of solutions [9]. In this case, we seek a solution vector $\delta\mathbf{m}$ such that the Euclidean norm of the residual vector $\|[A]\delta\mathbf{m} - \mathbf{r}\|$ is minimized. The resultant solution is referred to as the least-squares solution to equation (24). If the least-squares problem has more than one solution, the one having the minimum Euclidean norm is called the minimum-norm solution. Because the analytical and the actual mass matrices are presumed to be close, then if $\delta\mathbf{m}$ has an infinite number of solutions, the minimum-norm solution $\delta\mathbf{m}$ will be used to update the analytical mass matrix.

At first glance it appears that one needs to solve an underdetermined, least-squares problem of size $N_e^2 \times N^2$ (assuming $N_e < N$) in order to update the mass matrix of the system. However, the optimal matrix storage scheme commonly used in finite elements [10] can be utilized to pass along the sparsity information, thereby imposing the condition that all the zero elements in the analytical mass matrix remain zeros in the adjusted mass matrix to drastically reduce the size of the problem to be solved. Mathematically, this can be achieved by eliminating all the zero elements from $\delta\mathbf{m}$ and by deleting all the corresponding columns in $[A]$. For example, if the actual mass matrix of the system is known to be diagonal, then $\delta m_{ij} = 0$ for $i \neq j$, and equation (24) reduces to

$$[A']\,\delta\mathbf{m}' = \mathbf{r}, \tag{27}$$

where

$$\delta\mathbf{m}' = [\delta m_{11} \;\; \delta m_{22} \;\cdots\; \delta m_{NN}]^{\mathrm{T}}. \tag{28}$$

Thus, the initial $N_e^2 \times N^2$ underdetermined, least-squares problem is reduced to one of size $N_e^2 \times N$. The resulting least-squares problem will be either overdetermined (that is, the number of equations is greater than the number of unknowns) or underdetermined, depending on whether $N_e^2 \geqslant N$ or $N_e^2 < N$ respectively. For most physical systems, the number of measured modes will generally be substantially less than the size of the analytical model, and the problem will be underdetermined.

Once the mass matrix has been corrected, then equation (12) can be used to update the stiffness matrix as

$$[X]^{\mathrm{T}}[\delta K][X] = ([I'] + [X]^{\mathrm{T}}[\delta M][X])[\Lambda] - [\Lambda']. \tag{29}$$

Equation (29) can also be expanded so that $[\delta K]$ appears as an unknown column vector, leading to yet another underdetermined, least-squares problem of the form

$$[B]\,\delta\mathbf{k} = \mathbf{h}. \tag{30}$$

Enforcing the connectivity information of the stiffness matrix, we can drastically reduce the size of the least-squares problem to be solved. For example, if the stiffness matrix of the system is known to be tri-diagonal, then $\delta k_{ij} = 0$ for $|i - j| > 1$, and equation (30) reduces to

$$[B']\,\delta\mathbf{k}' = \mathbf{h}, \tag{31}$$

where $[B']$ is obtained from $[B]$ by deleting all the appropriate columns and

$$\delta\mathbf{k}' = [\delta k_{11}\ \ \delta k_{12}\,|\,\delta k_{21}\ \ \delta k_{22}\ \ \delta k_{23}\,|\ \cdots|\,\delta k_{NN-1}\ \ \delta k_{NN}]^{\mathrm{T}}. \tag{32}$$

Thus, the initial $N_e^2 \times N^2$ underdetermined, least-squares problem is reduced to one of size $N_e^2 \times (3N - 2)$.

Finally, a few words about the connectivity information are warranted. Because the basis of model updating is the analytical model, the analytical model itself must reflect the actual system to a certain degree. Here, we assume that the analytical model and the actual system share the same sparsity pattern. The structural mass and stiffness parameters of the analytical and actual systems, however, may differ substantially. For systems whose connectivity information is now well known, we can use engineering judgement to estimate the zero and non-zero patterns in the mass and stiffness matrices, and delete the appropriate elements from the mass and stiffness correction vectors in the solution of the least-squares problem. Terms that we think might be non-zero are included in the analysis. Using an iterative scheme, we can isolate the non-zero mass and stiffness correction terms. Preliminary numerical simulations suggest that convergence to the correct sparsity pattern is usually achieved within a few iterations even for a limited number of measured modes. How to apply the updating scheme iteratively to correct the system parameters will be addressed in a future paper.

3. RESULTS

We now apply the various model updating algorithms to the simple system of Figure 1, whose mass matrix is diagonal and whose stiffness matrix is symmetric and tri-diagonal.

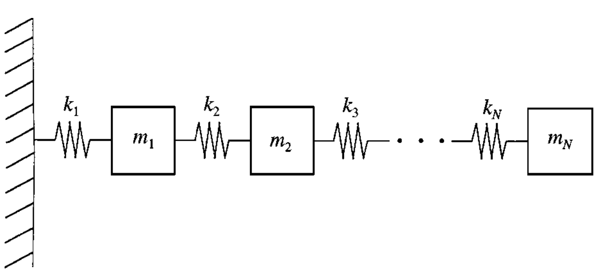

Figure 1. Simple chain of coupled oscillators.

TABLE 1

The actual and the updated masses (kg), obtained by using the Lagrange multipliers formalism [3], the perturbation approach [8], and the proposed (new) mass updating scheme, for $N_e = N = 16$. The analytical masses are 2·000 kg

m_{actual}	$m_{Lagrange}$	$m_{perturbation}$	m_{new}
m_1 = 2·01278	1·73342	2·66997	2·01295
m_2 = 2·54431	2·17979	3·11584	2·54459
m_3 = 2·50904	2·16914	3·51100	2·50919
m_4 = 2·82288	2·42666	3·19962	2·82293
m_5 = 2·64104	2·29568	3·45863	2·64105
m_6 = 1·97303	1·75149	3·48676	1·97304
m_7 = 2·55730	2·19251	3·77886	2·55730
m_8 = 2·32015	2·05055	3·97162	2·32016
m_9 = 2·63741	2·51566	4·46901	2·63741
m_{10} = 1·29419	1·49824	3·23780	1·29418
m_{11} = 2·18310	2·19574	2·92685	2·18311
m_{12} = 1·31170	1·41313	2·45586	1·31170
m_{13} = 2·65806	2·38878	2·97832	2·65806
m_{14} = 2·43710	2·28836	3·01630	2·43710
m_{15} = 1·76514	1·76796	2·94783	1·76514
m_{16} = 2·85021	2·50437	2·94283	2·85022

The analytical masses and stiffnesses are 2·000 kg and 5·000 N/m, respectively. The primary objective here is to compare the results of the proposed algorithm with the Lagrange multipliers formalism and the perturbation scheme. Thus, such a simple system is sufficient. Knowing the modes of vibration of the analytical model and the actual system, we aim to correct the analytical system matrices.

Finally, when solving a least-square problem, the CMLIB routine *sglss* was accessed, which is specialized to handle both underdetermined and overdetermined systems $[A]\mathbf{x} = \mathbf{b}$, where $[A]$ is an $m \times n$ matrix and $\mathbf{b}$ is a vector of length m. When the system is overdetermined ($m \geqslant n$), the least-square solution is computed by decomposing the matrix $[A]$ into the product of an orthogonal matrix $[Q]$ and an upper triangular matrix $[R]$ (QR fractorization). When the system is underdetermined ($m < n$), the minimal length solution is computed by factoring the matrix $[A]$ into the product of a lower triangular matrix $[L]$ and an orthogonal matrix $[Q]$ (LQ factorization). If the matrix $[A]$ is determined to be rank deficient, that is the rank of $[A]$ is less than $\min(m, n)$, then the minimal length least-squares solution is computed.

Table 1 shows the actual and the updated mass parameters for the system of Figure 1, with $N = 16$ and $N_e = 16$. To perform the mass updating algorithm, three lumped masses of magnitude 0·200 kg are added to masses 5, 10 and 15. Note that the added masses are an order of magnitude smaller than the nominal analytical masses. The Lagrange multipliers formalism of equation (1) return 11 masses that deviate by over 10·00% from their actual values. Because the deviations between the actual and the analytical masses are large, the perturbation approach fails and equation (5) returns 13 masses that deviate by over 20·00% from their actual values. The proposed mass updating algorithm of equation (24) is by far the best. In fact, it corrects all the masses to within 0·01% of their actual values. Finally, of the three mass updating schemes, only the proposed algorithm returns a mass matrix that is strictly diagonal (the Lagrange multipliers method and the perturbation approach return fully populated mass matrices, though it should be noted that the off-diagonal terms are at

TABLE 2

The actual and the updated stiffnesses (N/m), obtained by using the Lagrange multipliers formalism [5], *the perturbation approach* [8], *and the proposed (new) stiffness updating scheme, for* $N_e = N = 16$. *The analytical stiffness are* 5·000 *N/m*

k_{actual}	$k_{Lagrange}$	$k_{perturbation}$	k_{new}
k_1 = 4·13999	3·60478	4·70009	4·14024
k_2 = 6·88016	5·87737	9·86087	6·88082
k_3 = 5·60515	4·70799	11·4754	5·60563
k_4 = 6·51076	5·66840	13·0297	6·51102
k_5 = 2·93431	2·57516	10·0978	2·93434
k_6 = 7·13261	6·54563	15·6957	7·13263
k_7 = 3·30715	2·96001	14·6169	3·30716
k_8 = 3·29861	2·95419	15·5731	3·29862
k_9 = 6·20207	6·38906	15·0097	6·20207
k_{10} = 6·63989	9·21278	14·4228	6·63989
k_{11} = 5·94890	8·05651	7·27202	5·94891
k_{12} = 6·32030	8·26645	9·89750	6·32030
k_{13} = 3·33570	4·20197	6·56427	3·33570
k_{14} = 5·98769	6·20317	7·57026	5·98770
k_{15} = 5·49729	6·61043	9·70472	5·49729
k_{16} = 5·94830	6·19617	9·42443	5·94830

least one order of magnitude less than the diagonal components), because it is the only procedure that allows the connectivity information to be easily enforced.

Table 2 displays the updated stiffness parameters. By inspection, note that the perturbation approach of updating the stiffness values (see equation (7)) is the worst, resulting in 8 stiffness that deviate by over 100·00% from their actual values. The Lagrange multipliers formalism is also unacceptable, returning 5 stiffness parameters that deviate by over 20·00% from their actual values. The proposed method of updating the stiffnesses (see equation (30)), on the other hand, returns stiffness values that are all with 0·01% of the actual stiffness. Moreover, while the Lagrange multipliers method and the perturbation approach lead to full stiffness matrices, the new stiffness updating scheme returns a tri-diagonal stiffness matrix, thus preserving the physical load paths of the analytical system.

Table 3 shows the eigenvalues (the square of the natural frequencies) of the system obtained by using the three distinct model updating methods. The original analytical eigenvalues are also listed for comparison. Note that the Lagrange multipliers formalism leads to updated eigenvalues that are identical to those obtained experimentally, even though the resulting mass and stiffness matrices deviate substantially from the exact (see Tables 1 and 2). This is, however, not surprising, because the experimental eigendata are used as constraints that are explicitly satisfied in the updating algorithm [5]. The perturbation scheme leads to an updated model whose eigenvalues are all within 11·28% of the exact values, despite the large deviations in the updated masses and stiffnesses from the actual values. Finally, the proposed algorithm leads to an updated model whose eigenvalues are all with 0·01% of the measured data.

We now consider the more realistic case where the test data is incomplete. Specifically, we consider the case of $N_e = 4$, i.e., only the first 4 modes of vibrations can be measured. Table 4 shows the adjusted mass parameters obtained by using the various updating methods. By inspection, while the Lagrange multipliers approach and the perturbation

TABLE 3

The analytical, actual and the updated eigenvalues (1/s^2), *obtained by using the Lagrange multipliers formalism* [3,5], *the perturbation approach* [8], *and the proposed* (*new*) *mass/stiffness updating schemes, for* $N_e = N = 16$

$\lambda_{analytical}$	λ_{actual}	$\lambda_{Lagrange}$	$\lambda_{perturbation}$	λ_{new}
$\lambda_1 = 0{\cdot}02264$	0·01950	0·01950	0·01808	0·01950
$\lambda_2 = 0{\cdot}20254$	0·17220	0·17220	0·16526	0·17220
$\lambda_3 = 0{\cdot}55582$	0·43171	0·43171	0·41017	0·43170
$\lambda_4 = 1{\cdot}06973$	0·92643	0·92643	0·82198	0·92642
$\lambda_5 = 1{\cdot}72570$	1·41718	1·41718	1·48868	1·41718
$\lambda_6 = 2{\cdot}50000$	2·13323	2·13323	2·09934	2·13323
$\lambda_7 = 3{\cdot}36466$	2·69188	2·69188	2·77164	2·69188
$\lambda_8 = 4{\cdot}28843$	3·28285	3·28285	3·60767	3·28285
$\lambda_9 = 5{\cdot}23791$	5·21935	5·21935	5·20351	5·21935
$\lambda_{10} = 6{\cdot}17879$	6·21042	6·21042	6·22527	6·21042
$\lambda_{11} = 7{\cdot}07708$	6·35818	6·35818	6·76697	6·35818
$\lambda_{12} = 7{\cdot}90028$	8·12766	8·12766	8·08535	8·12767
$\lambda_{13} = 8{\cdot}61867$	9·06829	9·06829	8·92026	9·06829
$\lambda_{14} = 9{\cdot}20627$	9·19906	9·19906	9·65551	9·19906
$\lambda_{15} = 9{\cdot}64184$	9·87109	9·87109	10·8315	9·87109
$\lambda_{16} = 9{\cdot}90964$	13·5781	13·5781	14·2722	13·5781

TABLE 4

The actual and the updated masses (*kg*), *obtained by using the Lagrange multipliers formalism* [3], *the perturbation approach* [8], *and the proposed* (*new*) *mass updating scheme, for* $N_e = 4$. *The analytical masses are* 2·000 *kg*

m_{actual}	$m_{Lagrange}$	$m_{perturbation}$	m_{new}
$m_1 = 2{\cdot}01278$	2·02300	2·04672	2·42061
$m_2 = 2{\cdot}54431$	2·04685	2·13077	2·25814
$m_3 = 2{\cdot}50904$	2·05725	2·15938	2·58772
$m_4 = 2{\cdot}82288$	2·04467	2·12269	2·81276
$m_5 = 2{\cdot}64104$	2·01050	2·09176	2·63390
$m_6 = 1{\cdot}97303$	2·01187	2·10848	1·98473
$m_7 = 2{\cdot}55730$	2·01693	2·13097	2·54469
$m_8 = 2{\cdot}32015$	1·98706	2·10563	2·33686
$m_9 = 2{\cdot}63741$	1·96165	2·04907	2·62558
$m_{10} = 1{\cdot}29419$	1·94682	2·01949	1·29676
$m_{11} = 2{\cdot}18310$	1·94086	2·03112	2·18014
$m_{12} = 1{\cdot}31170$	1·95412	2·04514	1·31533
$m_{13} = 2{\cdot}65806$	1·99045	2·04359	2·65673
$m_{14} = 2{\cdot}43710$	2·01084	2·06182	2·43810
$m_{15} = 1{\cdot}76514$	2·02694	2·12485	1·76388
$m_{16} = 2{\cdot}85021$	2·03523	2·18928	2·85090

method return updated masses that are merely perturbations of the analytical masses, the proposed algorithm (see equation (24)) yields updated masses that are much closer to the actual values than either of the other two schemes. In fact, except for the first two masses, the proposed approach returns masses that are all within 3·14% of the exact values, despite the fact that only a fourth of the total modes are used to perform the update.

TABLE 5

The actual and the updated stiffnesses (N/m), obtained by using the Lagrange multipliers formalism [5], the perturbation approach [8], and the proposed (new) stiffness updating scheme, and the hybrid approach, for $N_e = 4$. The analytical stiffness are 5·000 N/m

k_{actual}	$k_{Lagrange}$	$k_{perturbation}$	k_{new}	k_{hybrid}
k_1 = 4·13999	4·54229	5·01397	4·46771	4·97859
k_2 = 6·88016	5·13440	4·99187	5·30249	8·27362
k_3 = 5·60515	4·75973	4·99386	5·30779	5·46921
k_4 = 6·51076	5·39377	5·00787	4·97209	6·51719
k_5 = 2·93431	4·32750	5·00977	3·18452	2·93377
k_6 = 7·13261	5·49889	4·98627	5·95659	7·13899
k_7 = 3·30715	5·00384	4·96212	3·72830	3·30643
k_8 = 3·29861	4·80610	4·96976	4·01147	3·29345
k_9 = 6·20207	5·12207	5·00125	5·37398	6·19543
k_{10} = 6·63989	4·95199	5·01509	5·41229	6·63673
k_{11} = 5·94890	4·86396	4·99283	4·96171	5·94447
k_{12} = 6·32030	5·07264	4·96483	4·69794	6·31748
k_{13} = 3·33570	4·79146	4·96716	5·05936	3·33579
k_{14} = 5·98769	5·16347	4·99206	5·12066	5·98729
k_{15} = 5·49729	4·94268	5·00286	5·16572	5·49695
k_{16} = 5·94830	4·95825	4·98885	4·95487	5·94949

TABLE 6

The analytical, actual and the updated eigenvalues ($1/s^2$), obtained by using the Lagrange multipliers formalism [3,5], the perturbation approach [8], and the proposed (new) mass/stiffness updating schemes, and the hybrid approach, for $N_e = 4$

λ_{actual}	$\lambda_{Lagrange}$	$\lambda_{perturbation}$	λ_{new}	λ_{hybrid}
λ_1 = 0·01950	0·01950	0·01945	0·09542	0·01985
λ_2 = 0·17220	0·17220	0·17514	0·13407	0·17223
λ_3 = 0·43171	0·43171	0·44122	0·35350	0·43170
λ_4 = 0·92643	0·92643	0·96532	0·80168	0·92632
λ_5 = 1·41718	1·65880	1·72570	1·26314	1·41620
λ_6 = 2·13323	2·49296	2·50000	2·24535	2·13047
λ_7 = 2·69188	3·33621	3·36466	2·71693	2·66496
λ_8 = 3·28285	4·26473	4·28843	3·45927	3·27800
λ_9 = 5·21935	5·23740	5·23791	4·51129	5·21657
λ_{10} = 6·21042	6·15514	6·17879	5·02477	6·18390
λ_{11} = 6·35818	7·05988	7·07708	5·97026	6·33734
λ_{12} = 8·12766	7·87660	7·90028	7·21494	8·13369
λ_{13} = 9·06829	8·59786	8·61867	7·53572	9·19861
λ_{14} = 9·19906	9·20411	9·20627	8·09895	9·85985
λ_{15} = 9·87109	9·59884	9·64184	9·44835	9·86910
λ_{16} = 13·5781	9·87893	9·90964	11·3674	13·5623

Table 5 displays the updated stiffness parameters. By inspection, no approach performs any better than the others. Table 6 shows the eigenvalues of the updated models. Not surprisingly, because the first four measured modes of vibration are used as constraints [5], the Lagrange multipliers formalism exactly reproduces the first four eigenvalues of the actual system. Interestingly, except for the first four updated eigenvalues, the updated eigenvalues obtained by the perturbation approach are all identical to those of the analytical system. Because the proposed stiffness updating algorithm returns adjusted stiffnesses that deviate substantially from the actual stiffnesses, the agreement between the actual and the resulting eigenvalues obtained by the proposed mass/stiffness updating scheme is poor. Finally, the previous examples reveal that the accuracy of all the updating algorithms depends on the number of measured modes, N_e, that are available for analysis. The more experimental data one uses in the updating algorithm the more accurate the updated model becomes.

From various numerical experiments, the proposed mass updating scheme returns a mass matrix that closely resembles the actual system mass matrix, even for a limited number of measured modes. Unfortunately, none of the stiffness updating algorithms used thus far can correct the stiffness matrix to the same level of accuracy when N_e is small. To remedy the situation, we turn to Kabe's approach [6], which assumes the mass matrix to be known. Thus, we apply the following hybrid approach: we first update the system mass matrix using the added mass approach; we then apply Kabe's procedure which preserves the physical connectivity of the system to update the stiffness matrix. In essence, we harness the advantages of the mass updating algorithm introduced in this paper and Kabe's approach of correcting the stiffness matrix. Table 5 also shows the updated stiffness parameters obtained by using this hybrid scheme. Note that except for the first three stiffness parameters, the remaining updated stiffnesses are all within 0·16% of the actual stiffness values (the third updated stiffness is within 2·45% of the actual value). Table 6 also includes the updated eigenvalues obtained by the hybrid approach. Note how well they track the measured data (except for λ_{14}, the remaining updated eigenvalues are all with 1·45% of the measured eigenvalues), despite the fact that only for measured modes of vibration are used to update the system mass and stiffness matrices. Because Kabe's algorithm is very computationally intensive, a trade-off has to be made regarding accuracy versus computational efficiency.

We now turn our attention to the modal matrices obtained by the various updating techniques. Given any two modal matrices, some terms in one modal matrix are closer to the exact values than the other while other terms are farther. Thus, it is difficult to pass judgement on which updating algorithm leads to a more accurate solution by simple inspection of the modal matrices. It is common practice to check for the correctness of the modal matrix by resorting to be orthogonality characteristics of the normal modes. If the modal matrix $[X]$, is exact and properly normalized, then it is orthogonal with respect to the actual mass matrix $[M]$, such that

$$[X]^{\mathrm{T}}[M][X] = [I], \tag{33}$$

where $[I]$ is the identity. Because the updated modal matrices are approximate, performing the above triple product yields

$$[X_u]^{\mathrm{T}}[M][X_u] = [I_u], \tag{34}$$

where $[X_u]$ is the updated modal matrix, normalized such that the diagonal elements of equation (34) are identically one, and $[I_u]$ is a full matrix. Because the updated modal matrices are not orthogonal with respect to $[M]$, the average magnitude of the off-diagonal terms of equation (34) can be used to describe the accuracy of the modal matrices quantitatively. Table 7 shows the average magnitudes of the off-diagonal terms of equation (34). When the set of the measured modes is complete ($N_e = N = 16$), the Lagrange multipliers scheme and the proposed mass/stiffness algorithm return modal matrices that are exact. For $N_e = 4$, the proposed mass/stiffness updating method yields a modal matrix that is closer to the exact than either the Lagrange multipliers formalism or the perturbation

TABLE 7

The average magnitude of the off-diagonal terms of equation (34). *If the modal matrix is exact, then the average is identically zero*

Approach (N_e)	Average
Lagrange multipliers (16)	0·00000
Perturbation (16)	0·05276
Proposed mass/stiffness (16)	0·00000
Lagrange multipliers (4)	0·04235
Perturbation (4)	0·04302
Proposed mass/stiffness (4)	0·00719
Hybrid (4)	0·00063

method. The hybrid updating scheme (which consists of the proposed mass updating algorithm and Kabe's approach) returns a modal matrix that is even better than the proposed mass/stiffness updating algorithm.

Finally, only natural frequencies and mode shapes are used here to correct the system mass and stiffness matrices. While the Lagrange multipliers formalism leads to an updated model whose analytical eigendata coincide exactly with those obtained experimentally (for $N_e = N$), it yields a model whose system parameters may deviate substantially from the actual values (see Tables 1 and 2). Thus, the ability to duplicate the free response characteristics may be insufficient to guarantee the updated model to be useful or accurate. To improve our confidence level in the modified analytical model, more information may be required in the updating scheme, including using multiple boundary conditions [11] or using anti-resonance data [12] to increase the available experimental data. In essence, the more constraints the updated model can satisfy, the more accurate it is in describing the actual system.

4. CONCLUSION

New mass and stiffness updating algorithms were presented. Because of their simplicity, the known connectivity information can be easily imposed, thus preserving the physical configuration of the analytical model and reducing the amount of computational efforts required to correct the analytical system matrices. When the set of measured data is complete, the newly developed model updating schemes return mass and stiffness matrices that are exact. When the set of experimental modes is incomplete, the proposed mass updating algorithm still yields a mass matrix that is nearly correct. In conjunction with Kabe's stiffness updating formalism, which assumes the mass matrix to be known, a hybrid approach is introduced that can be used to accurately update finite element models with limited experimental data. While the proposed updating scheme requires more work and cause testing down-time, the additional time and effort are a relatively small price to pay for the ability to correct the analytical model.

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